CK-12

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CHAPTER 2

CHAPTER 2 Reasoning and Proof

Chapter Outline

This chapter explains how to reason and how to use reasoning to prove theorems about angle pairs and segments. This chapter also introduces the properties of congruence, which will also be used in proofs. Subsequent chapters will combine what you have learned in Chapters 1 and 2 and build upon them.

2.1 Conditional Statements

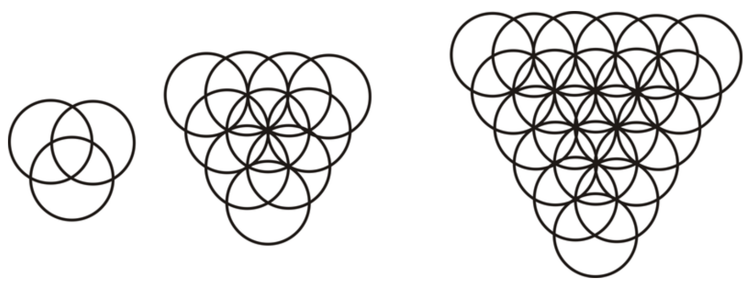
Learning Objectives

- Identify the hypothesis and conclusion of an if-then or conditional statement.
- Write the converse, inverse, and contrapositive of an if-then statement.
- Recognize a biconditional statement.

Review Queue

Find the next figure or term in the pattern.

a. 5, 8, 12, 17, 23,...
b. $\frac{2}{5}, \frac{3}{6}, \frac{4}{7}, \frac{5}{9}, \frac{6}{10}, \ldots$

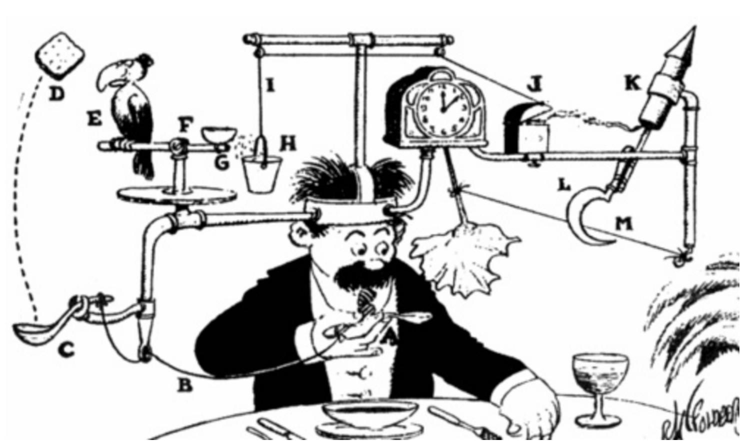


c.

d. Find a counterexample for the following conjectures.

   a. If it is April, then it is Spring Break.
   b. If it is June, then I am graduating.

Know What? Rube Goldman was a cartoonist in the 1940s who drew crazy inventions to do very simple things. The invention to the right has a series of smaller tasks that leads to the machine wiping the man's face with a napkin.



Write a series of if-then statements to that would caption this cartoon, from A to M.

If-Then Statements

Conditional Statement (also called an **If-Then Statement**): A statement with a hypothesis followed by a conclusion.

Another way to define a conditional statement is to say, "If this happens, then that will happen."

Hypothesis: The first, or "if," part of a conditional statement. An educated guess.

Conclusion: The second, or "then," part of a conditional statement. The conclusion is the result of a hypothesis.

Keep in mind that conditional statements might not always be written in the "if-then" form. Here are a few examples.

Statement 1: If you work overtime, then you'll be paid time-and-a-half.

Statement 2: I'll wash the car if the weather is nice.

Statement 3: If 2 divides evenly into x, then x is an even number.

Statement 4: I'll be a millionaire when I win monopoly.

Statement 5: All equiangular triangles are equilateral.

Statements 1 and 3 are written in the "if-then" form. The hypothesis of Statement 1 is "you work overtime." The conclusion is "you'll be paid time-and-a-half."

So, if Sarah works overtime, then what will happen? From Statement 1, we can conclude that she will be paid time-and-a-half.

If 2 goes evenly into 16, what can you conclude? From Statement 3, we know that 16 must be an even number.

Statement 2 has the hypothesis after the conclusion. Even though the word "then" is not there, the statement can be rewritten as: If the weather is nice, then I'll wash the car. If the word "if" is in the middle of a conditional statement, the hypothesis is always after it.

Statement 4 uses the word "when" instead of "if." It should be treated like Statement 2, so it can be written as: If I win monopoly, then I will be a millionaire.

Statement 5 "if" and "then" are not there, but can be rewritten as: If a triangle is equiangular, then it is equilateral.

Converse, Inverse, and Contrapositive of a Conditional Statement

Look at **Statement 2** again: *If the weather is nice, then I'll wash the car.*

This can be rewritten using letters to represent the hypothesis and conclusion.

$$\begin{aligned} \text{If } p, \text{then } q. \qquad p &= \text{the weather is nice} \\ q &= \text{I'll wash the car} \\ \text{Or, } p &\rightarrow q \end{aligned}$$

In addition to these positives, we can also write the negations, or "not"s of p and q. The symbolic version of not p, is $\sim p$.

$\sim p =$ the weather is not nice
$\sim q =$ I won't wash the car

Using these negations and switching the order of p and q, we can create three more conditional statements.

Converse	$q \to p$	$\underbrace{\text{If I wash the car}}_{q}, \underbrace{\text{then the weather is nice}}_{p}.$
Inverse	$\sim p \to \sim q$	$\underbrace{\text{If the weather is not nice}}_{\sim p}, \underbrace{\text{then I won't wash the car}}_{\sim q}.$
Contrapositive	$\sim q \to \sim p$	$\underbrace{\text{If I don't wash the car}}_{\sim q}, \underbrace{\text{then the weather is not nice}}_{\sim p}.$

If we accept "If the weather is nice, then I'll wash the car" as true, then the converse and inverse are not necessarily true. However, if we take original statement to be true, then the contrapositive is also true. We say that the contrapositive is ***logically equivalent*** to the original if-then statement.

Example 1: Use the statement: If $n > 2$, then $n^2 > 4$.

a) Find the converse, inverse, and contrapositive.

b) Determine if the statements from part a are true or false. If they are false, find a counterexample.

Solution: The original statement is true.

Converse :	If $n^2 > 4$, then $n > 2$.	*False*. n could be -3, making $n^2 = 9$.
Inverse :	If $n < 2$, then $n^2 < 4$.	*False*. Again, if $n = -3$, then $n^2 = 9$.
Contrapositive :	If $n^2 < 4$, then $n < 2$.	*True*, the only square number less than 4 is 1, which has square roots of 1 or -1, both less than 2.

Example 2: Use the statement: If I am at Disneyland, then I am in California.

a) Find the converse, inverse, and contrapositive.

b) Determine if the statements from part a are true or false. If they are false, find a counterexample.

Solution: The original statement is true.

Converse :	If I am in California, then I am at Disneyland. *False*. I could be in San Francisco.
Inverse :	If I am not at Disneyland, then I am not in California. *False*. Again, I could be in San Francisco.
Contrapositive :	If I am not in California, then I am not at Disneyland. *True*. If I am not in the state, I couldn't be at Disneyland.

Notice for the inverse and converse ***we can use the same counterexample.*** This is because the inverse and converse are also ***logically equivalent.***

Example 3: Use the statement: Any two points are collinear.

a) Find the converse, inverse, and contrapositive.

b) Determine if the statements from part a are true or false. If they are false, find a counterexample.

Solution: First, change the statement into an "if-then" statement: If two points are on the same line, then they are collinear.

Converse :	If two points are collinear, then they are on the same line. $True$.
Inverse :	If two points are not on the same line, then they are not collinear. $True$.
Contrapositive :	If two points are not collinear, then they do not lie on the same line. $True$.

Biconditional Statements

Example 3 is an example of a biconditional statement.

Biconditional Statement: When the original statement and converse are both true.

So, $p \to q$ is true and $q \to p$ is true. It is written $p \leftrightarrow q$, with a double arrow to indicate that it does not matter if p or q is first. It is said, "p if and only if q"

Example 4: Rewrite Example 3 as a biconditional statement.

Solution: *If two points are on the same line, then they are collinear* can be rewritten as: *Two points are on the same line if and only if they are collinear.*

Replace the "if-then" with "if and only if" in the middle of the statement. "If and only if" can be abbreviated "iff."

Example 5: The following is a true statement:

$m\angle ABC > 90^\circ$ if and only if $\angle ABC$ is an obtuse angle.

Determine the two true statements within this biconditional.

Solution:

Statement 1: If $m\angle ABC > 90^\circ$, then $\angle ABC$ is an obtuse angle

Statement 2: If $\angle ABC$ is an obtuse angle, then $m\angle ABC > 90^\circ$.

You should recognize this as the definition of an obtuse angle. All geometric definitions are biconditional statements.

Example 6: $p : x < 10 \qquad q : 2x < 50$

a) Is $p \to q$ true? If not, find a counterexample.

b) Is $q \to p$ true? If not, find a counterexample.

c) Is $\sim p \to \sim q$ true? If not, find a counterexample.

d) Is $\sim q \to \sim p$ true? If not, find a counterexample.

Solution:

$$amp; p \to q : \quad \text{If } x < 10, \text{ then } 2x < 50. \quad True.$$
$$amp; q \to p : \quad \text{If } 2x < 50, \text{ then } x < 10. \quad False, \ x = 15 \text{ would be a counterexample.}$$
$$\sim p \to \sim q : \quad \text{If } x > 10, \text{ then } 2x > 50. \quad False, \ x = 15 \text{ would also work here.}$$
$$\sim q \to \sim p : \quad \text{If } 2x > 50, \text{ then } x > 10. \quad True.$$

Know What? Revisited The conditional statements are as follows:

$A \to B$: If the man raises his spoon, then it pulls a string.

$B \to C$: If the string is pulled, then it tugs back a spoon.

$C \to D$: If the spoon is tugged back, then it throws a cracker into the air.

$D \to E$: If the cracker is tossed into the air, the bird will eat it.

$E \to F$: If the bird eats the cracker, then it turns the pedestal.

$F \to G$: If the bird turns the pedestal, then the water tips over.

$G \to H$: If the water tips over, it goes into the bucket.

$H \to I$: If the water goes into the bucket, then it pulls down the string.

$I \to J$: If the bucket pulls down the string, then the string opens the box.

$J \to K$: If the box is opened, then a fire lights the rocket.

$K \to L$: If the rocket is lit, then the hook pulls a string.

$L \to M$: If the hook pulls the string, then the man's faces is wiped with the napkin.

This is a very complicated contraption used to wipe a man's face. Purdue University liked these cartoons so much, that they started the Rube Goldberg Contest in 1949. This past year, the task was to pump hand sanitizer into someone's hand in no less than 20 steps. http://www.purdue.edu/newsroom/rubegoldberg/index.html

Review Questions

For questions 1-6, determine the hypothesis and the conclusion.

1. If 5 divides evenly into x, then x ends in 0 or 5.
2. If a triangle has three congruent sides, it is an equilateral triangle.
3. Three points are coplanar if they all lie in the same plane.
4. If $x = 3$, then $x^2 = 9$.
5. If you take yoga, then you are relaxed.
6. All baseball players wear hats.
7. Write the converse, inverse, and contrapositive of #1. Determine if they are true or false. If they are false, find a counterexample.
8. Write the converse, inverse, and contrapositive of #5. Determine if they are true or false. If they are false, find a counterexample.
9. Write the converse, inverse, and contrapositive of #6. Determine if they are true or false. If they are false, find a counterexample.
10. Find the converse of #2. If it is true, write the biconditional of the statement.
11. Find the converse of #3. If it is true, write the biconditional of the statement.
12. Find the converse of #4. If it is true, write the biconditional of the statement.

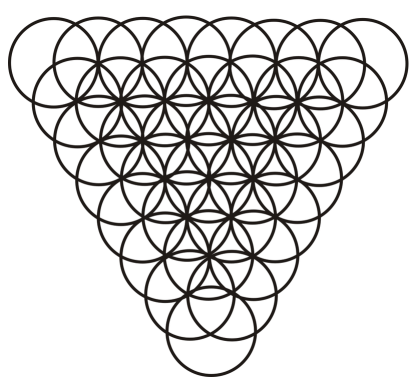
For questions 13-16, use the statement: If $AB = 5$ and $BC = 5$, then B is the midpoint of $\overline{AC}$.

13. If this is the converse, what is the original statement? Is it true?
14. If this is the original statement, what is the inverse? Is it true?
15. Find a counterexample of the statement.
16. Find the contrapositive of the original statement from #13.
17. What is the inverse of the inverse of $p \to q$? HINT: Two wrongs make a right in math!
18. What is the one-word name for the converse of the inverse of an if-then statement?
19. What is the one-word name for the inverse of the converse of an if-then statement?
20. What is the contrapositive of the contrapositive of an if-then statement?

For questions 21-24, determine the two true conditional statements from the given biconditional statements.

21. A U.S. citizen can vote if and only if he or she is 18 or more years old.
22. A whole number is prime if and only if it has exactly two distinct factors.
23. Points are collinear if and only if there is a line that contains the points.
24. $2x = 18$ if and only if $x = 9$.
25. $p : x = 4 \quad q : x^2 = 16$
    a. Is $p \to q$ true? If not, find a counterexample.
    b. Is $q \to p$ true? If not, find a counterexample.
    c. Is $\sim p \to \sim q$ true? If not, find a counterexample.
    d. Is $\sim q \to \sim p$ true? If not, find a counterexample.
26. $p : x = -2 \quad q : -x + 3 = 5$
    a. Is $p \to q$ true? If not, find a counterexample.
    b. Is $q \to p$ true? If not, find a counterexample.
    c. Is $\sim p \to \sim q$ true? If not, find a counterexample.
    d. Is $\sim q \to \sim p$ true? If not, find a counterexample.
27. p : the measure of $\angle ABC = 90^\circ$ q : $\angle ABC$ is a right angle
    a. Is $p \to q$ true? If not, find a counterexample.
    b. Is $q \to p$ true? If not, find a counterexample.
    c. Is $\sim p \to \sim q$ true? If not, find a counterexample.
    d. Is $\sim q \to \sim p$ true? If not, find a counterexample.
28. p : the measure of $\angle ABC = 45^\circ$ q : $\angle ABC$ is an acute angle
    a. Is $p \to q$ true? If not, find a counterexample.
    b. Is $q \to p$ true? If not, find a counterexample.
    c. Is $\sim p \to \sim q$ true? If not, find a counterexample.
    d. Is $\sim q \to \sim p$ true? If not, find a counterexample.
29. Write a conditional statement. Write the converse, inverse and contrapositive of your statement. Are they true or false? If they are false, write a counterexample.
30. Write a true biconditional statement. Separate it into the two true conditional statements.

Review Queue Answers

a. 30
b. $\frac{7}{11}$
c. 

   a. It could be another day that isn't during Spring Break. Spring Break doesn't last the entire month.
   b. You could be a freshman, sophomore or junior. There are several counterexamples.

2.2 Inductive Reasoning

Learning Objectives

- Recognize visual and number patterns.
- Extend and generalize patterns.
- Write a counterexample.

Review Queue

a. Look at the patterns of numbers below. Determine the next three numbers in the list. Describe the pattern.
   a. 1, 2, 3, 4, 5, 6, _____, _____, _____
   b. 3, 6, 9, 12, 15, _____, _____, _____
   c. 1, 4, 9, 16, 25, _____, _____, _____
b. Are the statements below true or false? If they are false, state why.
   a. Perpendicular lines form four right angles.
   b. Angles that are congruent are also equal.
   c. Linear pairs are always congruent.
c. For the line, $y = 3x + 1$, make an $x - y$ table for $x = 1, 2, 3, 4$, and 5. What do you notice? How does it relate to $1b$?

Know What? This is the "famous" locker problem:

A new high school has just been completed. There are 1000 lockers in the school and they have been numbered from 1 through 1000. During recess, the students decide to try an experiment. When recess is over each student walks into the school one at a time. The first student will open all of the locker doors. The second student will close all of the locker doors with even numbers. The third student will change all of the locker doors that are multiples of 3 (*change means closing lockers that are open, and opening lockers that are closed*). The fourth student will change the position of all locker doors numbered with multiples of four and so on.

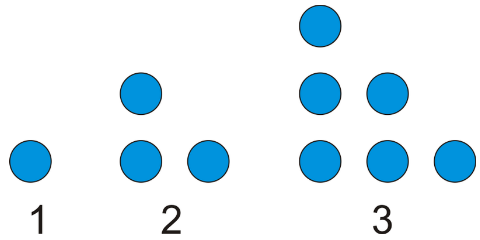
Imagine that this continues until the 1000 students have followed the pattern with the 1000 lockers. At the end, which lockers will be open and which will be closed? Which lockers were touched the most often? Which lockers were touched exactly 5 times?

Visual Patterns

Inductive Reasoning: Making conclusions based upon observations and patterns.

Let's look at some visual patterns to get a feel for what inductive reasoning is.

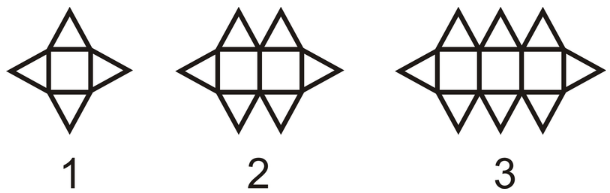
Example 1: A dot pattern is shown below. How many dots would there be in the bottom row of the 4^{th} figure? What would the *total number* of dots be in the 6^{th} figure?



Solution: There will be 4 dots in the bottom row of the 4^{th} figure. There is one more dot in the bottom row of each figure than in the previous figure.

There would be a total of 21 dots in the 6^{th} figure, $6+5+4+3+2+1$.

Example 2: How many *triangles* would be in the 10^{th} figure?



Solution: There are 10 squares, with a triangle above and below each square. There is also a triangle on each end of the figure. That makes $10+10+2=22$ triangles in all.

Example 2b: If one of these figures contains 34 triangles, how many *squares* would be in that figure?

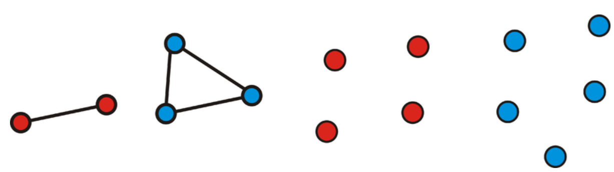
Solution: First, the pattern has a triangle on each end. Subtracting 2, we have 32 triangles. Now, divide 32 by 2 because there is a row of triangles above and below each square. $32 \div 2 = 16$ squares.

Example 2c: How can we find the number of triangles if we know the figure number?

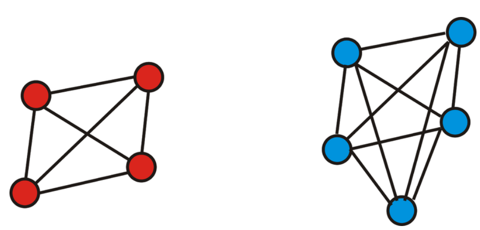
Solution: Let n be the figure number. This is also the number of squares. $2n$ is the number of triangles above and below the squares. Add 2 for the triangles on the ends.

If the figure number is n, then there are $2n+2$ triangles in all.

Example 3: For two points, there is one line segment between them. For three non-collinear points, there are three line segments with those points as endpoints. For four points, no three points being collinear, how many line segments are between them? If you add a fifth point, how many line segments are between the five points?



Solution: Draw a picture of each and count the segments.



For 4 points there are 6 line segments and for 5 points there are 10 line segments.

Number Patterns

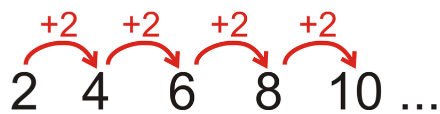
Let's look at a few examples.

Example 4: Look at the pattern 2, 4, 6, 8, 10,...

a) What is the 19^{th} term in the pattern?

b) Describe the pattern and try and find an equation that works for every term in the pattern.

Solution: For part a, each term is 2 more than the previous term.



You could count out the pattern until the 19^{th} term, but that could take a while. The easier way is to recognize the pattern. Notice that the 1^{st} term is $2 \cdot 1$, the 2^{nd} term is $2 \cdot 2$, the 3^{rd} term is $2 \cdot 3$, and so on. So, the 19^{th} term would be $2 \cdot 19$ or 38.

For part b, we can use this pattern to generate a formula. Typically with number patterns we use n to represent the term number. So, this pattern is 2 times the term number, or $2n$.

Example 5: Look at the pattern 1, 3, 5, 7, 9, 11,...

a) What is the 34^{th} term in the pattern?

b) What is the n^{th} term?

Solution: The pattern increases by 2 and is odd. From the previous example, we know that if a pattern increases by 2, you would multiply n by 2. However, this pattern is odd, so we need to add or subtract a number. Let's put what we know into a table:

TABLE 2.1:

n	$2n$	-1	**Pattern**
1	2	-1	1
2	4	-1	3
3	6	-1	5
4	8	-1	7
5	10	-1	9
6	12	-1	11

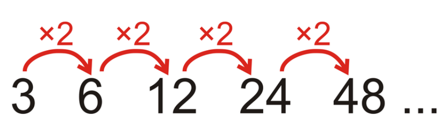
From this we can reason that the 34^{th} term would be $34 \cdot 2$ minus 1, which is 67. Therefore, the n^{th} term would be $2n - 1$.

Example 6: Look at the pattern: 3, 6, 12, 24, 48,...

a) What is the next term in the pattern? The 10^{th} term?

b) Make a rule for the n^{th} term.

Solution: This pattern is different than the previous two examples. Here, each term is multiplied by 2 to get the next term.



Therefore, the next term will be $48 \cdot 2$ or 96. To find the 10^{th} term, we need to work on the pattern, let's break apart each term into the factors to see if we can find the rule.

TABLE 2.2:

n	**Pattern**	**Factors**	**Simplify**
1	3	3	$3 \cdot 2^0$
2	6	$3 \cdot 2$	$3 \cdot 2^1$
3	12	$3 \cdot 2 \cdot 2$	$3 \cdot 2^2$
4	48	$3 \cdot 2 \cdot 2 \cdot 2$	$3 \cdot 2^3$
5	48	$3 \cdot 2 \cdot 2 \cdot 2 \cdot 2$	$3 \cdot 2^4$

Using this equation, the 10^{th} term will be $3 \cdot 2^9$, or 1536. Notice that the exponent is one less than the term number. So, for the n^{th} term, the equation would be $3 \cdot 2^{n-1}$.

Example 7: Find the 8^{th} term in the list of numbers as well as the rule.

$$2, \frac{3}{4}, \frac{4}{9}, \frac{5}{16}, \frac{6}{25} \dots$$

Solution: First, change 2 into a fraction, or $\frac{2}{1}$. So, the pattern is now $\frac{2}{1}, \frac{3}{4}, \frac{4}{9}, \frac{5}{16}, \frac{6}{25} \dots$ Separate the top and the bottom of the fractions into two different patterns. The top is 2, 3, 4, 5, 6. It increases by 1 each time, so the 8^{th} term's numerator is 9. The denominators are the square numbers, so the 8^{th} term's denominator is 10^2 or 100. Therefore, the 8^{th} term is $\frac{9}{100}$. The rule for this pattern is $\frac{n+1}{n^2}$.

To summarize:

- If the same number is **added** from one term to the next, then you multiply n by it.
- If the same number is **multiplied** from one term to the next, then you would multiply the first term by increasing powers of this number. n or $n-1$ is in the exponent of the rule.
- If the pattern has **fractions**, separate the numerator and denominator into two different patterns. Find the rule for each separately.

Conjectures and Counterexamples

Conjecture: An "educated guess" that is based on examples in a pattern.

Numerous examples may make you believe a conjecture. However, no number of examples can actually *prove* a conjecture. It is always possible that the next example would show that the conjecture is false.

Example 8: Here's an algebraic equation and a table of values for n and with the result for t.

$$t = (n-1)(n-2)(n-3)$$

TABLE 2.3:

n	$(n-1)(n-2)(n-3)$	t
1	$(0)(-1)(-2)$	0
2	$(1)(0)(-1)$	0
3	$(2)(1)(0)$	0

After looking at the table, Pablo makes this conjecture:

The value of $(n-1)(n-2)(n-3)$ *is 0 for any whole number value of n.*

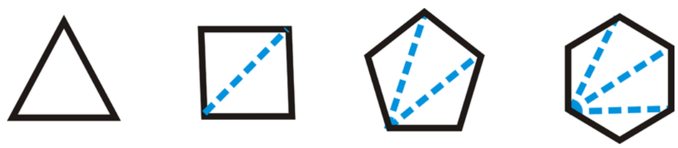
Is this a valid, or true, conjecture?

Solution: No, this is not a valid conjecture. If Pablo were to continue the table to $n=4$, he would have see that $(n-1)(n-2)(n-3)=(4-1)(4-2)(4-3)=(3)(2)(1)=6$.

In this example $n=4$ is called a counterexample.

Counterexample: An example that disproves a conjecture.

Example 9: Arthur is making figures for a graphic art project. He drew polygons and some of their diagonals.



Based on these examples, Arthur made this conjecture:

If a convex polygon has n sides, then there are $n-3$ triangles drawn from any given vertex of the polygon.

Is Arthur's conjecture correct? Can you find a counterexample to the conjecture?

Solution: The conjecture appears to be correct. If Arthur draws other polygons, in every case he will be able to draw $n-3$ triangles if the polygon has n sides.

Notice that we have ***not proved*** Arthur's conjecture, but only found several examples that hold true. This type of conjecture would need to be proven by induction.

Know What? Revisited Start by looking at the pattern. Red numbers are OPEN lockers.

Student 1 changes every locker:

1, 2, 3, 4, 5, 6, 7, 8,... 1000

Student 2 changes every 2^{nd} locker:

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12,... 1000

Student 3 changes every 3^{rd} locker:

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12,... 1000

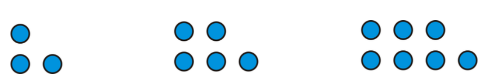
Student 4 changes every 4^{th} locker:

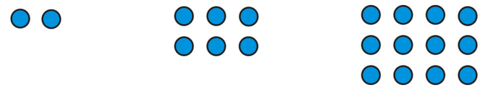
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12,... 1000

If you continue on in this way, the only lockers that will be left open are the numbers with an odd number of factors, or the square numbers: 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, 256, 289, 324, 361, 400, 441, 484, 529, 576, 625, 676, 729, 784, 841, 900, and 961. The lockers that were touched the most are the numbers with the most factors. The one locker that was touched the most was 840, which has 32 factors and thus, touched 32 times. There are three lockers that were touched exactly five times: 16, 81, and 625.

Review Questions

For questions 1 and 2, determine how many dots there would be in the 4^{th} and the 10^{th} pattern of each figure below.

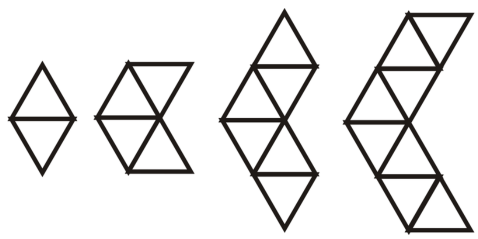
1. 

2. 

3. Use the pattern below to answer the questions.

   a. Draw the next figure in the pattern.
   b. How does the number of points in each star relate to the figure number?
   c. Use part *b* to determine a formula for the n^{th} figure.

4. Use the pattern below to answer the questions. All the triangles are equilateral triangles.



   a. Draw the next figure in the pattern. How many triangles does it have?
   b. Determine how many triangles are in the 24^{th} figure.
   c. How many triangles are in the n^{th} figure?

For questions 5-12, determine: 1) the next two terms in the pattern, 2) the 35^{th} figure and 3) the formula for the n^{th} figure.

5. 5, 8, 11, 14, 17,...
6. 6, 1, -4, -9, -14,...
7. 2, 4, 8, 16, 32,...
8. 67, 56, 45, 34, 23,...
9. 9, -4, 6, -8, 3,...
10. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots$
11. $\frac{2}{3}, \frac{4}{7}, \frac{6}{11}, \frac{8}{15}, \frac{10}{19}, \ldots$
12. 3, -5, 7, -9, 11,...
13. -1, 5, -9, 13, -17,...
14. $\frac{-1}{2}, \frac{1}{4}, \frac{-1}{6}, \frac{1}{8}, \frac{-1}{10}, \ldots$
15. 5, 12, 7, 10, 9,...
16. 1, 4, 9, 16, 25,...

For questions 13-16, determine the next two terms and describe the pattern.

17. 3, 6, 11, 18, 27,...
18. 3, 8, 15, 24, 35,...
19. 1, 8, 27, 64, 125,...
20. 1, 1, 2, 3, 5,...

We all use inductive reasoning in our daily lives. The process consists of making observations, recognizing a pattern and making a generalization or conjecture. Read the following examples of reasoning in the real world and determine if they are examples of Inductive reasoning. Do you think the conjectures are true or can you give a counterexample?

21. For the last three days Tommy has gone for a walk in the woods near his house at the same ime of day. Each time he has seen at least one deer. Tommy reasons that if he goes for a walk tomorrow at the same time, he will see deer again.
22. Maddie likes to bake. She especially likes to take recipes and make substitutions to try to make them healthier. She might substitute applesauce for sugar or oat flour for white flour. She has noticed that she needs to add more baking powder or baking soda than the recipe indicates in these situations in order for the baked goods to rise appropriately.
23. One evening Juan saw a chipmunk in his backyard. He decided to leave a slice of bread with peanut butter on it for the creature to eat. The next morning the bread was gone. Juan concluded that chipmunks like to eat bread with peanut butter.
24. Describe an instance in your life when either you or someone you know used inductive reasoning to correctly make a conclusion.
25. Describe an instance when you observed someone using invalid reasoning skills.

Challenge For the following patterns find a) the next two terms, b) the 40^{th} term and c) the n^{th} term rule. You will need to think about each of these in a different way. *Hint: Double all the values and look for a pattern in their factors. Once you come up with the rule remember to divide it by two to undo the doubling.*

26. 2, 5, 9, 14,...
27. 3, 6, 10, 15,...
28. 3, 12, 30, 60,...

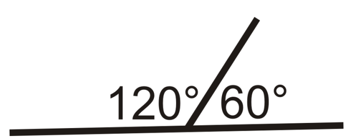
Connections to Algebra

29. Plot the values of the terms in the sequence 3, 8, 13,... against the term numbers in the coordinate plane. In other words, plot the points (1, 3), (2, 8), and (3, 13). What do you notice? Could you use algebra to figure out the "rule" or equation which maps each term number (x) to the correct term value (y)? Try it.
30. Which sequences in problems 5-16 follow a similar pattern to the one you discovered in #29? Can you use inductive reasoning to make a conclusion about which sequences follow the same type of rule?

Review Queue Answers

a. 7, 8, 9
b. 18, 21, 24
c. 36, 49, 64

a. true
b. true
c. false,



2.3 Deductive Reasoning

Learning Objectives

- Apply some basic rules of logic.
- Compare inductive reasoning and deductive reasoning.
- Use truth tables to analyze patterns of logic.

Review Queue

1. Write the converse, inverse, and contrapositive of the following statement:

Football players wear shoulder pads.

2. Is the converse, inverse or contrapositive of #1 true? If not, find a counterexample.

3. If flowers are in bloom, then it is spring.

If it is spring, then the weather is nice.

So, if flowers are blooming, what can we conclude?

Know What? In a fictitious far-away land, a poor peasant is awaiting his fate from the king. He is standing in a stadium, filled with spectators pointing and wondering what is going to happen. Finally, the king directs everyone's attention to two doors, at the floor level with the peasant. Both doors have signs on them, which are below:

TABLE 2.4:

Door A	Door B
IN THIS ROOM THERE IS A LADY, AND IN THE OTHER ROOM THERE IS A TIGER.	IN ONE OF THESE ROOMS THERE IS A LADY, AND IN ONE OF THE OTHER ROOMS THERE IS A TIGER.

The king states, "Only one of these statements is true. If you pick correctly, you will find the lady. If not, the tiger will be waiting for you." Which door should the peasant pick?

Deductive Reasoning

Logic: The study of reasoning.

In the first section, you learned about inductive reasoning, which is to make conclusions based upon patterns and observations. Now, we will learn about deductive reasoning. Deductive reasoning draws conclusions from facts.

Deductive Reasoning: When a conclusion is drawn from facts. Typically, conclusions are drawn from general statements about something more specific.

Example 1: Suppose Bea makes the following statements, which are known to be true.

If Central High School wins today, they will go to the regional tournament.

Central High School won today.

What is the logical conclusion?

Solution: This is an example of deductive reasoning. There is one logical conclusion if these two statements are true: *Central High School will go to the regional tournament.*

Example 2: Here are two true statements.

Every odd number is the sum of an even and an odd number.

5 is an odd number.

What can you conclude?

Solution: Based on only these two true statements, there is one conclusion: *5 is the sum of an even and an odd number.* (This is true, $5 = 3+2$ or $4+1$).

Law of Detachment

Let's look at Example 2 and change it into symbolic form.

$p:$ A number is odd $\quad q:$ It is the sum of an even and odd number

So, the first statement is $p \rightarrow q$.

- The second statement in Example 2, "5 is an odd number," is a specific example of p. "A number" is 5.
- The conclusion is q. Again it is a specific example, such as $4+1$ or $2+3$.

The symbolic form of Example 2 is:

$$\begin{aligned} & p \rightarrow q \\ & p \\ & \therefore q \qquad \therefore \text{ symbol for "therefore"} \end{aligned}$$

All deductive arguments that follow this pattern have a special name, the Law of Detachment.

Law of Detachment: Suppose that $p \rightarrow q$ is a true statement and given p. Then, you can conclude q.

Another way to say the Law of Detachment is: "If $p \rightarrow q$ is true, and p is true, then q is true."

Example 3: Here are two true statements.

If $\angle A$ and $\angle B$ are a linear pair, then $m\angle A + m\angle B = 180^\circ$.

$\angle ABC$ and $\angle CBD$ are a linear pair.

What conclusion can you draw from this?

Solution: This is an example of the Law of Detachment, therefore:

$$m\angle ABC + m\angle CBD = 180^\circ$$

Example 4: Here are two true statements. *Be careful!*

If $\angle A$ *and* $\angle B$ *are a linear pair, then* $m\angle A + m\angle B = 180^\circ$.

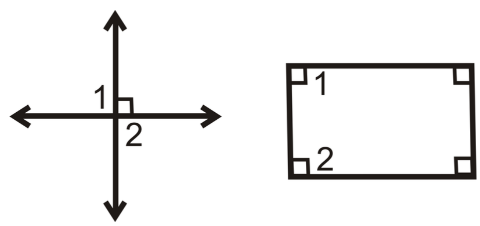
$m\angle 1 = 90^\circ$ *and* $m\angle 2 = 90^\circ$.

What conclusion can you draw from these two statements?

Solution: Here there is NO conclusion. These statements are in the form:

$$p \to q$$
$$q$$

We ***cannot*** conclude that $\angle 1$ and $\angle 2$ are a linear pair. We are told that $m\angle 1 = 90^\circ$ and $m\angle 2 = 90^\circ$ and while $90^\circ + 90^\circ = 180^\circ$, this does not mean they are a linear pair. Here are two counterexamples.



In both of these counterexamples, $\angle 1$ and $\angle 2$ are right angles. In the first, they are vertical angles and in the second, they are two angles in a rectangle.

This is called the *Converse Error* because the second statement is the conclusion of the first, like the converse of a statement.

Law of Contrapositive

Example 5: The following two statements are true.

If a student is in Geometry, then he or she has passed Algebra I.

Daniel has not passed Algebra I.

What can you conclude from these two statements?

Solution: These statements are in the form:

$$p \to q$$
$$\sim q$$

Not q is the beginning of the contrapositive $(\sim q \to \sim p)$, therefore the logical conclusion is *not p: Daniel is not in Geometry.*

This example is called the Law of Contrapositive.

Law of Contrapositive: Suppose that $p \to q$ is a true statement and given $\sim q$. Then, you can conclude $\sim p$.

Recall that the logical equivalent to a conditional statement is its contrapositive. Therefore, the Law of Contrapositive is a logical argument.

Example 6: Determine the conclusion from the true statements below.

Babies wear diapers.

My little brother does not wear diapers.

Solution: The second statement is the equivalent of $\sim q$. Therefore, the conclusion is $\sim p$, or: *My little brother is not a baby.*

Example 7a: Determine the conclusion from the true statements below.

If you are not in Chicago, then you can't be on the L.

Bill is in Chicago.

Solution: If we were to rewrite this symbolically, it would look like:

$$\begin{aligned} &\sim p \rightarrow \sim q \\ &p \end{aligned}$$

This is not in the form of the Law of Contrapositive or the Law of Detachment, so there is no logical conclusion. You cannot conclude that Bill is on the *L* because he could be anywhere in Chicago. This is an example of the *Inverse Error* because the second statement is the negation of the hypothesis, like the beginning of the inverse of a statement.

Example 7b: Determine the conclusion from the true statements below.

If you are not in Chicago, then you can't be on the L.

Sally is on the L.

Solution: If we were to rewrite this symbolically, it would look like:

$$\begin{aligned} &\sim p \rightarrow \sim q \\ &q \end{aligned}$$

Even though it looks a little different, this is an example of the Law of Contrapositive. Therefore, the logical conclusion is: *Sally is in Chicago.*

Law of Syllogism

Example 8: Determine the conclusion from the following true statements.

If Pete is late, Mark will be late.

If Mark is late, Karl will be late.

So, if Pete is late, what will happen?

Solution: If Pete is late, this starts a domino effect of lateness. Mark will be late and Karl will be late too. So, if Pete is late, then *Karl will be late*, is the logical conclusion.

Each "then" becomes the next "if" in a chain of statements. The chain can consist of any number of connected statements. This is called the Law of Syllogism

Law of Syllogism: If $p \rightarrow q$ and $q \rightarrow r$ are true, then $p \rightarrow r$ is the logical conclusion.

Typically, when there are more than two linked statements, we continue to use the next letter(s) in the alphabet to represent the next statement(s); $r \rightarrow s, s \rightarrow t$, and so on.

Example 9: Look back at the **Know What? Revisited** from the previous section. There were 12 linked if-then statements, making one LARGE Law of Syllogism. Write the conclusion from these statements.

Solution: Symbolically, the statements look like this:

$A \to B$	$B \to C$	$C \to D$	$D \to E$	$E \to F$	$F \to G$
$G \to H$	$H \to I$	$I \to J$	$J \to K$	$K \to L$	$L \to M$
$\therefore A \to M$					

So, *If the man raises his spoon, then his face is wiped with the napkin.*

Inductive vs. Deductive Reasoning

You have now worked with both inductive and deductive reasoning. They are different but not opposites. Inductive reasoning means reasoning from examples or patterns. Enough examples might make you suspect that a relationship is always true. But, until you go beyond the inductive stage, you can't be absolutely sure that it is always true. That is, you cannot ***prove*** something is true with inductive reasoning.

That's where deductive reasoning takes over. Let's say we have a conjecture that was arrived at inductively, but is not proven. We can use the Law of Detachment, Law of Contrapositive, Law of Syllogism, and other logic rules to prove this conjecture.

Example 10: Determine if the following statements are examples of inductive or deductive reasoning.

a) Solving an equation for x.

b) 1, 10, 100, 1000,...

c) Doing an experiment and writing a hypothesis.

Solution: Inductive Reasoning = Patterns, Deductive Reasoning = Logic from Facts

a) Deductive Reasoning: Each step follows from the next.

b) Inductive Reasoning: This is a pattern.

c) Inductive Reasoning: You make a hypothesis or conjecture comes from the patterns that you found in the experiment (not facts). If you were to *prove* your hypothesis, then you would have to use deductive reasoning.

Truth Tables

So far we know these symbols for logic:

$\sim$ not (negation)

$\to$ if-then

$\therefore$ therefore

Two more symbols are:

$\wedge$ and

$\vee$ or

We would write "p and q" as $p \wedge q$ and "p or q" as $p \vee q$.

Truth tables use these symbols and are another way to analyze logic.

First, let's relate p and $\sim p$. To make it easier, set p as: *An even number.*

Therefore, $\sim p$ is *An odd number*. Make a truth table to find out if they are both true. Begin with all the "truths" of p, true (T) or false (F).

TABLE 2.5:

p
T
F

Next we write the corresponding truth values for $\sim p$. $\sim p$ has the opposite truth values of p. So, if p is true, then $\sim p$ is false and vise versa.

TABLE 2.6:

p	$\sim p$
T	F
F	T

Example 11: Draw a truth table for p, q and $p \wedge q$.

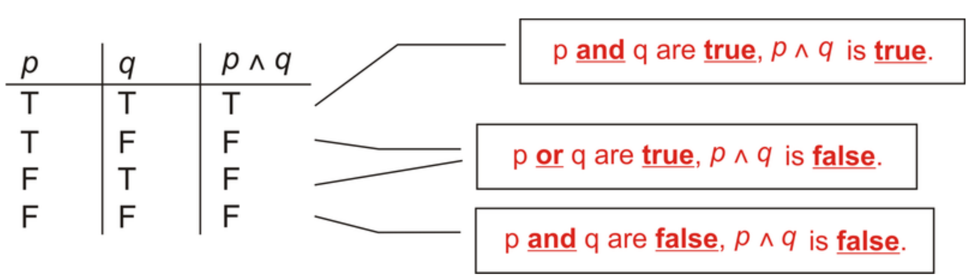
Solution: First, make columns for p and q. Fill the columns with all the possible true and false combinations for the two.

TABLE 2.7:

p	q
T	T
T	F
F	T
F	F

Notice all the combinations of p and q. **Anytime we have truth tables with two variables, this is always how we fill out the first two columns.**

Next, we need to figure out when $p \wedge q$ is true, based upon the first two columns. $p \wedge q$ **can only be true if BOTH** p **and** q **are true.** So, the completed table looks like this:



This is how a truth table with two variables and their "and" column is always filled out.

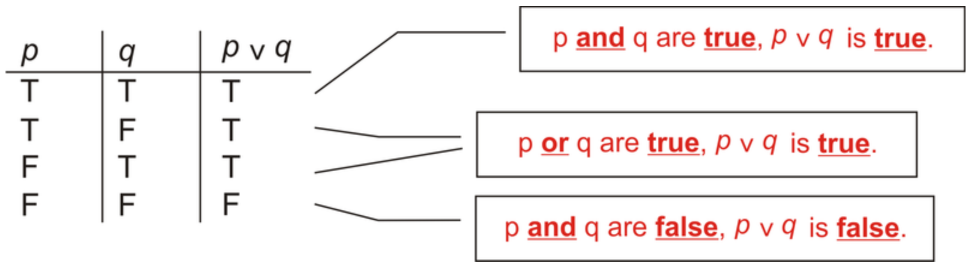
Example 12: Draw a truth table for p, q and $p \vee q$.

Solution: First, make columns for p and q, just like Example 11.

TABLE 2.8:

p	q
T	T
T	F
F	T
F	F

Next, we need to figure out when $p \vee q$ is true, based upon the first two columns. $p \vee q$ **is true if** p **OR** q **are true, or both are true.** So, the completed table looks like this:



The difference between $p \wedge q$ and $p \vee q$ is the second and third rows. For "and" both p and q have to be true, but for "or" only one has to be true.

Example 13: Determine the truths for $p \wedge (\sim q \vee r)$.

Solution: First, there are three variables, so we are going to need all the combinations of their truths. **For three variables, there are always 8 possible combinations.**

TABLE 2.9:

p	q	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

Next, address the $\sim q$. It will just be the opposites of the q column.

TABLE 2.10:

p	q	r	$\sim q$
T	T	T	F
T	T	F	F
T	F	T	T
T	F	F	T
F	T	T	F
F	T	F	F
F	F	T	T
F	F	F	T

Now, let's do what's in the parenthesis, $\sim q \vee r$. Remember, for "or" only $\sim q$ OR r has to be true. Only use the $\sim q$ and r columns to determine the values in this column.

TABLE 2.11:

p	q	r	$\sim q$	$\sim q \vee r$
T	T	T	F	T
T	T	F	F	F
T	F	T	T	T
T	F	F	T	T
F	T	T	F	T
F	T	F	F	F
F	F	T	T	T
F	F	F	T	T

Finally, we can address the entire problem, $p \wedge (\sim q \vee r)$. Use the p and $\sim q \vee r$ to determine the values. Remember, for "and" both p and $\sim q \vee r$ must be true.

TABLE 2.12:

p	q	r	$\sim q$	$\sim q \vee r$	$p \wedge (\sim q \vee r)$
T	T	T	F	T	T
T	T	F	F	F	F
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	F	T	F
F	T	F	F	F	F
F	F	T	T	T	F
F	F	F	T	T	F

To Recap:

- Start truth tables with all the possible combinations of truths. For 2 variables there are 4 combinations for 3 variables there are 8. **<u>You always start a truth table this way.</u>**
- Do any negations on the any of the variables.
- Do any combinations in parenthesis.
- Finish with completing what the problem was asking for.

Know What? Revisited Analyze the two statements on the doors.

<u>Door A</u>: IN THIS ROOM THERE IS A LADY, AND IN THE OTHER ROOM THERE IS A TIGER.

<u>Door B</u>: IN ONE OF THESE ROOMS THERE IS A LADY, AND IN ONE OF THE OTHER ROOMS THERE IS A TIGER.

We know that one door is true, so the other one must be false. Let's assume that Door A is true. That means the lady is behind Door A and the tiger is behind Door B. However, if we read Door B carefully, it says "in one of these rooms," which means the lady could be behind either door, which is actually the true statement. So, because Door B is the true statement, Door A is false and the tiger is actually behind it. Therefore, the peasant should pick Door B.

Review Questions

Determine the logical conclusion and state which law you used (Law of Detachment, Law of Contrapositive, or Law of Syllogism). If no conclusion can be drawn, write "no conclusion."

1. People who vote for Jane Wannabe are smart people. I voted for Jane Wannabe.
2. If Rae is the driver today then Maria is the driver tomorrow. Ann is the driver today.
3. If a shape is a circle, then it never ends. If it never ends, then it never starts. If it never starts, then it doesn't exist. If it doesn't exist, then we don't need to study it.
4. If you text while driving, then you are unsafe. You are a safe driver.
5. If you wear sunglasses, then it is sunny outside. You are wearing sunglasses.
6. If you wear sunglasses, then it is sunny outside. It is cloudy.
7. I will clean my room if my mom asks me to. I am not cleaning my room.
8. If I go to the park, I bring my dog. If I bring my dog, we play fetch with a stick. If we play fetch, my dog gets dirty. If my dog gets dirty, I give him a bath.
9. Write the symbolic representation of #3. Include your conclusion. Is this a sound argument? Does it make sense?
10. Write the symbolic representation of #1. Include your conclusion.
11. Write the symbolic representation of #7. Include your conclusion.

For questions 12 and 13, rearrange the order of the statements (you may need to use the Law of Contrapositive too) to discover the logical conclusion.

12. If I shop, then I will buy shoes. If I don't shop, then I didn't go to the mall. If I need a new watch battery, then I go to the mall.
13. If Anna's parents don't buy her ice cream, then she didn't get an A on her test. If Anna's teacher gives notes, Anna writes them down. If Anna didn't get an A on her test, then she couldn't do the homework. If Anna writes down the notes, she can do the homework.

Determine if the problems below represent inductive or deductive reasoning. Briefly explain your answer.

14. John is watching the weather. As the day goes on it gets more and more cloudy and cold. He concludes that it is going to rain.
15. Beth's 2-year-old sister only eats hot dogs, blueberries and yogurt. Beth decides to give her sister some yogurt because she is hungry.
16. Nolan Ryan has the most strikeouts of any pitcher in Major League Baseball. Jeff debates that he is the best pitcher of all-time for this reason.
17. Ocean currents and waves are dictated by the weather and the phase of the moon. Surfers use this information to determine when it is a good time to hit the water.
18. As Rich is driving along the 405, he notices that as he gets closer to LAX the traffic slows down. As he passes it, it speeds back up. He concludes that anytime he drives past an airport, the traffic will slow down.
19. Amani notices that the milk was left out on the counter. Amani remembers that she put it away after breakfast so it couldn't be her who left it out. She also remembers hearing her mother tell her brother on several occasions to put the milk back in the refrigerator. She concludes that he must have left it out.
20. At a crime scene, the DNA of four suspects is found to be present. However, three of them have an alibi for the time of the crime. The detectives conclude that the fourth possible suspect must have committed the crime.

Write a truth table for the following variables.

21. $p \wedge \sim p$

22. $\sim p \vee \sim q$
23. $p \wedge (q \vee \sim q)$
24. $(p \wedge q) \vee \sim r$
25. $p \vee (\sim q \vee r)$
26. $p \wedge (q \vee \sim r)$
27. The only difference between 19 and 21 is the placement of the parenthesis. How does the truth table differ?
28. When is $p \vee q \vee r$ true?

Is the following a valid argument? If so, what law is being used? HINT: Statements could be out of order.

29. $p \rightarrow q$
    $r \rightarrow p$
    $\therefore r \rightarrow q$
30. $p \rightarrow q$
    $r \rightarrow q$
    $\therefore p \rightarrow r$
31. $p \rightarrow \sim r$
    r
    $\therefore \sim p$
32. $\sim q \rightarrow r$
    q
    $\therefore \sim r$
33. $p \rightarrow (r \rightarrow s)$
    p
    $\therefore r \rightarrow s$
34. $r \rightarrow q$
    $r \rightarrow s$
    $\therefore q \rightarrow s$

Review Queue Answers

1. Converse: If you wear shoulder pads, then you are a football player.

Inverse: If you are not a football player, then you do not wear shoulder pads.

Contrapositive: If you do not wear shoulder pads, then you are not a football player.

2. The converse and inverse are both false. A counterexample for both could be a woman from the 80's. They definitely wore shoulder pads!

3. You could conclude that the weather is nice.

2.4 Algebraic and Congruence Properties

Learning Objectives

- Understand basic properties of equality and congruence.
- Solve equations and justify each step in the solution.
- Use a 2-column format to prove theorems.

Review Queue

Solve the following problems.

1. *Explain* how you would solve $2x-3=9$.

2. *If two angles are a linear pair, they are supplementary.*

If two angles are supplementary, their sum is 180°.

What can you conclude? By which law?

3. Draw a picture with the following:

$\angle LMN$ is bisected by $\overline{MO}$	$\overline{LM} \cong \overline{MP}$
$\angle OMP$ is bisected by $\overline{MN}$	N is the midpoint of $\overline{MQ}$

Know What? Three identical triplets are sitting next to each other. The oldest is Sara and she always tells the truth. The next oldest is Sue and she always lies. Sally is the youngest of the three. She sometimes lies and sometimes tells the truth.

Scott came over one day and didn't know who was who, so he asked each of them one question. Scott asked the sister that was sitting on the left, "Which sister is in the middle?" and the answer he received was, "That's Sara." Scott then asked the sister in the middle, "What is your name?" The response given was, "I'm Sally." Scott turned to the sister on the right and asked, "Who is in the middle?" The sister then replied, "She is Sue." Who was who?

Properties of Equality

Recall from Chapter 1 that the = sign and the word "equality" are used with numbers.

The basic properties of equality were introduced to you in Algebra I. Here they are again:

For all real numbers a, b, and c:

TABLE 2.13:

		Examples
Reflexive Property of Equality	$a=a$	$25=25$
Symmetric Property of Equality	$a=b$ and $b=a$	$m\angle P=90^\circ$ or $90^\circ=m\angle P$

TABLE 2.13: (continued)

		Examples
Transitive Property of Equality	$a = b$ and $b = c$, then $a = c$	$a+4 = 10$ and $10 = 6+4$, then $a+4 = 6+4$
Substitution Property of Equality	If $a = b$, then b can be used in place of a and vise versa.	If $a = 9$ and $a-c = 5$, then $9-c = 5$
Addition Property of Equality	If $a = b$, then $a+c = b+c$.	If $2x = 6$, then $2x+5 = 6+11$
Subtraction Property of Equality	If $a = b$, then $a-c = b-c$.	If $m\angle x+15^\circ = 65^\circ$, then $m\angle x+15^\circ-15^\circ = 65^\circ-15^\circ$
Multiplication Property of Equality	If $a = b$, then $ac = bc$.	If $y = 8$, then $5 \cdot y = 5 \cdot 8$
Division Property of Equality	If $a = b$, then $\frac{a}{c} = \frac{b}{c}$.	If $3b = 18$, then $\frac{3b}{3} = \frac{18}{3}$
Distributive Property	$a(b+c) = ab+ac$	$5(2x-7) = 5(2x)-5(7) = 10x-35$

Properties of Congruence

Recall that $\overline{AB} \cong \overline{CD}$ if and only if $AB = CD$. $\overline{AB}$ and $\overline{CD}$ represent segments, while AB and CD are lengths of those segments, which means that AB and CD are numbers. The properties of equality apply to AB and CD.

This also holds true for angles and their measures. $\angle ABC \cong \angle DEF$ if and only if $m\angle ABC = m\angle DEF$. Therefore, the properties of equality apply to $m\angle ABC$ and $m\angle DEF$.

Just like the properties of equality, there are properties of congruence. These properties hold for figures and shapes.

TABLE 2.14:

	For Line Segments	*For Angles*
Reflexive Property of Congruence	$\overline{AB} \cong \overline{AB}$	$\angle ABC \cong \angle CBA$
Symmetric Property of Congruence	If $\overline{AB} \cong \overline{CD}$, then $\overline{CD} \cong \overline{AB}$	If $\angle ABC \cong \angle DEF$, then $\angle DEF \cong \angle ABC$
Transitive Property of Congruence	If $\overline{AB} \cong \overline{CD}$ and $\overline{CD} \cong \overline{EF}$, then $\overline{AB} \cong \overline{EF}$	If $\angle ABC \cong \angle DEF$ and $\angle DEF \cong \angle GHI$, then $\angle ABC \cong \angle GHI$

Using Properties of Equality with Equations

When you solve equations in algebra you use properties of equality. You might not write out the logical justification for each step in your solution, but you should know that there is an equality property that justifies that step. We will abbreviate "Property of Equality" "PoE" and "Property of Congruence" "PoC."

Example 1: Solve $2(3x-4)+11 = x-27$ and justify each step.

Solution:

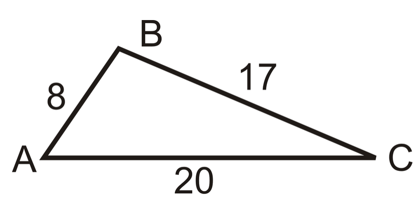
$$\begin{aligned}
2(3x-4)+11 &= x-27 & & \\
6x-8+11 &= x-27 & & \text{Distributive Property} \\
6x+3 &= x-27 & & \text{Combine like terms} \\
6x+3-3 &= x-27-3 & & \text{Subtraction PoE} \\
6x &= x-30 & & \text{Simplify} \\
6x-x &= x-x-30 & & \text{Subtraction PoE} \\
5x &= -30 & & \text{Simplify} \\
\frac{5x}{5} &= \frac{-30}{5} & & \text{Division PoE} \\
x &= -6 & & \text{Simplify}
\end{aligned}$$

Example 2: Given points A, B, and C, with $AB = 8, BC = 17$, and $AC = 20$. Are A, B, and C collinear?

Solution: Set up an equation using the Segment Addition Postulate.

$$\begin{aligned}
AB+BC &= AC & & \text{Segment Addition Postulate} \\
8+17 &= 20 & & \text{Substitution PoE} \\
25 &\neq 20 & & \text{Combine like terms}
\end{aligned}$$

Because the two sides are not equal, A, B and C are not collinear.



Example 3: If $m\angle A + m\angle B = 100^\circ$ and $m\angle B = 40^\circ$, prove that $\angle A$ is an acute angle.

Solution: We will use a 2-column format, with statements in one column and their corresponding reasons in the next. This is formally called a 2-column proof.

TABLE 2.15:

Statement	***Reason***
1. $m\angle A + m\angle B = 100^\circ$ and $m\angle B = 40^\circ$	***Given*** (always the reason for using facts that are told to us in the problem)
2. $m\angle A + 40^\circ = 100^\circ$	Substitution PoE
3. $m\angle A = 60^\circ$	Subtraction PoE
4. $\angle A$ is an acute angle	Definition of an acute angle, $m\angle A < 90^\circ$

Two-Column Proof

Example 4: Write a two-column proof for the following:

If A, B, C, and D are points on a line, in the given order, and $AB = CD$, then $AC = BD$.

Solution: First of all, when the statement is given in this way, the "if" part is the given and the "then" part is what we are trying to prove.

Always start with drawing a picture of what you are given.

Plot the points in the order A, B, C, D on a line.

Add the corresponding markings, $AB = CD$, to the line.

Draw the 2-column proof and start with the given information. From there, we can use deductive reasoning to reach the next statement and what we want to prove. **Reasons will be definitions, postulates, properties and previously proven theorems.**

TABLE 2.16:

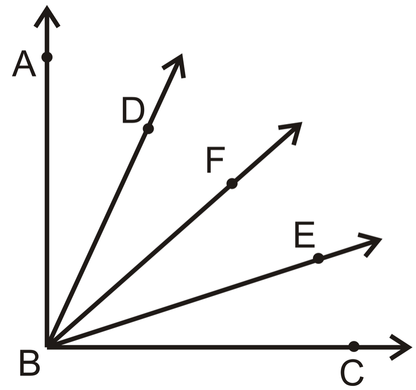
Statement	***Reason***
1. A, B, C, and D are collinear, in that order.	Given
2. $AB = CD$	Given
3. $BC = BC$	Reflexive PoE
4. $AB + BC = BC + CD$	Addition PoE
5. $AB + BC = AC$	
BC + CD = BD	Segment Addition Postulate
6. $AC = BD$	Substitution or Transitive PoE

When you reach what it is that you wanted to prove, you are done.

Prove Move: (A subsection that will help you with proofs throughout the book.) When completing a proof, a few things to keep in mind:

- **Number each step.**
- **Start with the given information.**
- **Statements with the same reason can (or cannot) be combined into one step.** It is up to you. For example, steps 1 and 2 above could have been one step. And, in step 5, the two statements could have been written separately.
- **Draw a picture and mark it with the given information.**
- **You must have a reason for EVERY statement.**
- **The order of the statements in the proof is not fixed.** For example, steps 3, 4, and 5 could have been interchanged and it would still make sense.

Example 5: Write a two-column proof.



Given: $\overrightarrow{BF}$ bisects $\angle ABC$; $\angle ABD \cong \angle CBE$

Prove: $\angle DBF \cong \angle EBF$

Solution: First, put the appropriate markings on the picture. Recall, that bisect means "to cut in half." Therefore, if $\overrightarrow{BF}$ bisects $\angle ABC$, then $m\angle ABF = m\angle FBC$. Also, because the word "bisect" was used in the given, the definition will probably be used in the proof.

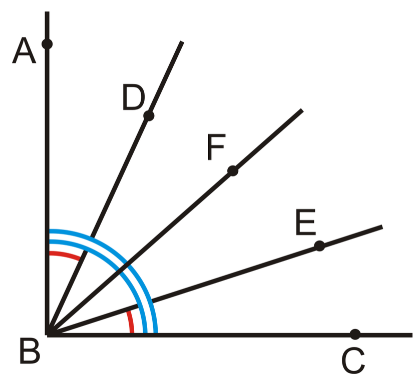


TABLE 2.17:

Statement	***Reason***
1. $\overrightarrow{BF}$ bisects $\angle ABC, \angle ABD \cong \angle CBE$	Given
2. $m\angle ABF = m\angle FBC$	Definition of an Angle Bisector
3. $m\angle ABD = m\angle CBE$	If angles are $\cong$, then their measures are equal.
4. $m\angle ABF = m\angle ABD + m\angle DBF$	
m $\angle FBC = m\angle EBF + m\angle CBE$	Angle Addition Postulate
5. $m\angle ABD + m\angle DBF = m\angle EBF + m\angle CBE$	Substitution PoE
6. $m\angle ABD + m\angle DBF = m\angle EBF + m\angle ABD$	Substitution PoE
7. $m\angle DBF = m\angle EBF$	Subtraction PoE
8. $\angle DBF \cong \angle EBF$	If measures are equal, the angles are $\cong$.

Prove Move: Use symbols and abbreviations for words within proofs. For example, $\cong$ was used in place of the word *congruent* above. You could also use $\angle$ for the word *angle*.

Know What? Revisited The sisters, in order are: Sally, Sue, Sara. The sister on the left couldn't have been Sara because that sister lied. The middle one could not be Sara for the same reason. So, the sister on the right must be Sara, which means she told Scott the truth and Sue is in the middle, leaving Sally to be the sister on the left.

Review Questions

For questions 1-8, solve each equation and justify each step.

1. $3x+11=-16$
2. $7x-3=3x-35$
3. $\frac{2}{3}g+1=19$
4. $\frac{1}{2}MN=5$
5. $5m\angle ABC=540^\circ$
6. $10b-2(b+3)=5b$
7. $\frac{1}{4}y+\frac{5}{6}=\frac{1}{3}$
8. $\frac{1}{4}AB+\frac{1}{3}AB=12+\frac{1}{2}AB$

For questions 9-14, use the given property or properties of equality to fill in the blank. x, y, and z are real numbers.

9. Symmetric: If $x=3$, then __________.
10. Distributive: If $4(3x-8)$, then __________.
11. Transitive: If $y=12$ and $x=y$, then __________.
12. Symmetric: If $x+y=y+z$, then __________.
13. Transitive: If $AB=5$ and $AB=CD$, then __________.
14. Substitution: If $x=y-7$ and $x=z+4$, then __________.
15. Given points E, F, and G and $EF=16$, $FG=7$ and $EG=23$. Determine if E, F and G are collinear.
16. Given points H, I and J and $HI=9, IJ=9$ and $HJ=16$. Are the three points collinear? Is I the midpoint?
17. If $m\angle KLM=56^\circ$ and $m\angle KLM+m\angle NOP=180^\circ$, explain how $\angle NOP$ must be an obtuse angle.

Fill in the blanks in the proofs below.

18. Given: $\angle ABC \cong DEF$
$\angle GHI \cong \angle JKL$ Prove: $m\angle ABC+m\angle GHI=m\angle DEF+m\angle JKL$

TABLE 2.18:

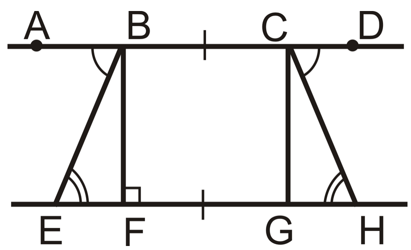
Statement	***Reason***
1.	Given
2. $m\angle ABC=m\angle DEF$ m $\angle GHI=m\angle JKL$	
3.	Addition PoE
4. $m\angle ABC+m\angle GHI=m\angle DEF+m\angle JKL$	

19. Given: M is the midpoint of $\overline{AN}$. N is the midpoint $\overline{MB}$ Prove: $AM=NB$

TABLE 2.19:

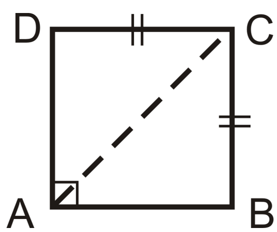
Statement	***Reason***
1.	Given
2.	Definition of a midpoint
3. $AM=NB$	

Use the diagram to answer questions 20-25.



20. Name a right angle.
21. Name two perpendicular lines.
22. Given that $EF = GH$, is $EG = FH$ true? Explain your answer.
23. Is $\angle CGH$ a right angle? Why or why not?
24. Using what is given in the picture AND $\angle EBF \cong \angle HCG$, prove $\angle ABF \cong \angle DCG$. Write a two-column proof.
25. Using what is given in the picture AND $AB = CD$, prove $AC = BD$. Write a two-column proof.

Use the diagram to answer questions 26-32.

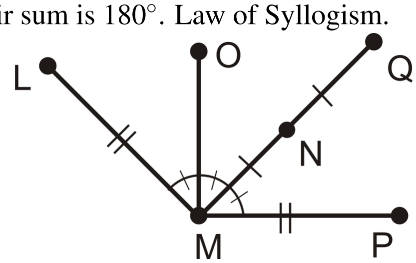


Which of the following must be true from the diagram?

Take each question separately, they do not build upon each other.

26. $\overline{AD} \cong \overline{BC}$
27. $\overline{AB} \cong \overline{CD}$
28. $\overline{CD} \cong \overline{BC}$
29. $\overline{AB} \perp \overline{AD}$
30. $ABCD$ is a square
31. $\overline{AC}$ bisects $\angle DAB$
32. Write a two-column proof. Given: Picture above and $\overline{AC}$ bisects $\angle DAB$Prove: $m\angle BAC = 45^\circ$
33. Draw a picture and write a two-column proof. Given: $\angle 1$ and $\angle 2$ form a linear pair and $m\angle 1 = m\angle 2$. Prove: $\angle 1$ is a right angle

Review Queue Answers

a. First, subtract 3 from both sides and then divide both sides by 2. $x = 3$
b. If 2 angles are a linear pair, then their sum is 180°. Law of Syllogism.
c. 

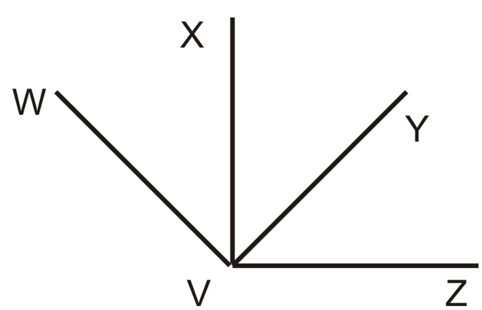
2.5 Proofs about Angle Pairs and Segments

Learning Objectives

- Use theorems about special pairs of angles.
- Use theorems about right angles and midpoints.

Review Queue

Write a 2-column proof

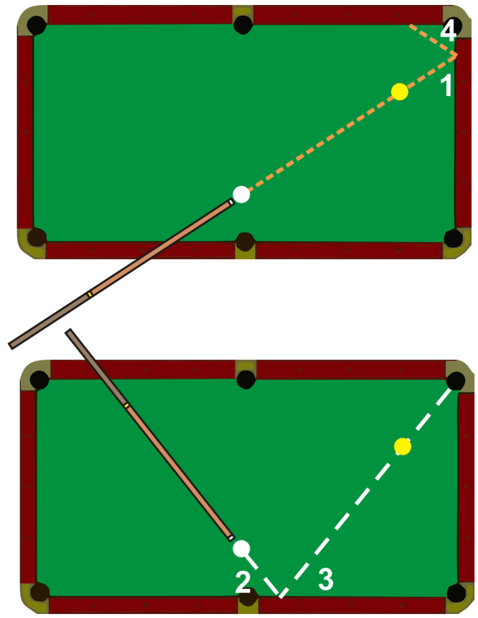


1. Given: $\overline{VX}$ is the angle bisector of $\angle WVY$.

$\overline{VY}$ is the angle bisector of $\angle XVZ$.

Prove: $\angle WVX \cong \angle YVZ$

Know What? The game of pool relies heavily on angles. The angle at which you hit the cue ball with your cue determines if a) you hit the yellow ball and b) if you can hit it into a pocket.

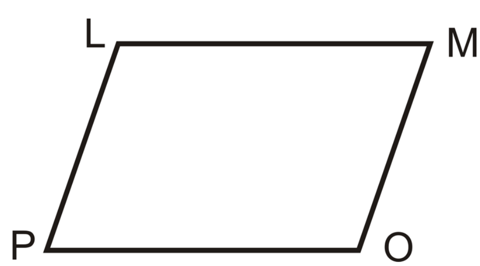


The top picture on the right illustrates if you were to hit the cue ball straight on and then hit the yellow ball. The orange line shows the path that the cue ball and then the yellow ball would take. You notice that $m\angle 1 = 56^\circ$. With a little focus, you notice that it makes more sense to approach the ball from the other side of the table and bank it off of the opposite side (see lower picture with the white path). You measure and need to hit the cue ball so that it hits the side of the table at a 50° angle (this would be $m\angle 2$). $\angle 3$ and $\angle 4$ are called the angles of reflection. Find the measures of these angles and how they relate to $\angle 1$ and $\angle 2$.

If you would like to play with the angles of pool, click the link for an interactive game. http://www.coolmath-games.com/0-poolgeometry/index.html

Naming Angles

As we learned in Chapter 1, angles can be addressed by numbers and three letters, where the letter in the middle is the vertex. We can shorten this label to one letter if there is only one angle with that vertex.



All of the angles in this parallelogram can be labeled by one letter, the vertex, instead of three.

$\angle MLP$ can be $\angle L$ $\angle LMO$ can be $\angle M$
$\angle MOP$ can be $\angle O$ $\angle OPL$ can be $\angle P$

This shortcut will now be used when applicable.

Right Angle Theorem: If two angles are right angles, then the angles are congruent.

Proof of the Right Angle Theorem

Given: $\angle A$ and $\angle B$ are right angles

Prove: $\angle A \cong \angle B$

TABLE 2.20:

Statement	*Reason*
1. $\angle A$ and $\angle B$ are right angles	Given
2. $m\angle A = 90^\circ$ and $m\angle B = 90^\circ$	Definition of right angles
3. $m\angle A = m\angle B$	Transitive PoE
4. $\angle A \cong \angle B$	$\cong$ angles have = measures

This theorem may seem redundant, but anytime right angles are mentioned, you need to use this theorem to say the angles are congruent.

Same Angle Supplements Theorem: If two angles are supplementary to the same angle then the angles are congruent.

So, if $m\angle A + m\angle B = 180^\circ$ and $m\angle C + m\angle B = 180^\circ$, then $m\angle A = m\angle C$. Using numbers to illustrate, we could say that if $\angle A$ is supplementary to an angle measuring 56°, then $m\angle A = 124^\circ$. $\angle C$ is also supplementary to 56°, so it too is 124°. Therefore, $m\angle A = m\angle C$. This example, however, does not constitute a proof.

Proof of the Same Angles Supplements Theorem

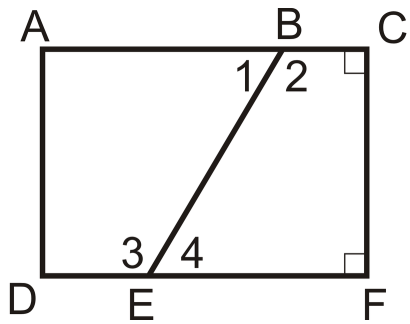
Given: $\angle A$ and $\angle B$ are supplementary angles. $\angle B$ and $\angle C$ are supplementary angles.

Prove: $\angle A \cong \angle C$

TABLE 2.21:

Statement	*Reason*
1. $\angle A$ and $\angle B$ are supplementary $\angle B$ and $\angle C$ are supplementary	Given
2. $m\angle A + m\angle B = 180^\circ$	
m $\angle B + m\angle C = 180^\circ$	Definition of supplementary angles
3. $m\angle A + m\angle B = m\angle B + m\angle C$	Substitution PoE
4. $m\angle A = m\angle C$	Subtraction PoE
5. $\angle A \cong \angle C$	$\cong$ angles have = measures

Example 1: Given that $\angle 1 \cong \angle 4$ and $\angle C$ and $\angle F$ are right angles, show which angles are congruent.



Solution: By the Right Angle Theorem, $\angle C \cong \angle F$. Also, $\angle 2 \cong \angle 3$ by the Same Angles Supplements Theorem. $\angle 1$ and $\angle 2$ are a linear pair, so they add up to 180°. $\angle 3$ and $\angle 4$ are also a linear pair and add up to 180°. Because $\angle 1 \cong \angle 4$, we can substitute $\angle 1$ in for $\angle 4$ and then $\angle 2$ and $\angle 3$ are supplementary to the same angle, making them congruent.

This is an example of a **paragraph proof**. Instead of organizing the proof in two columns, you explain everything in sentences.

Same Angle Complements Theorem: If two angles are complementary to the same angle then the angles are congruent.

So, if $m\angle A + m\angle B = 90^\circ$ and $m\angle C + m\angle B = 90^\circ$, then $m\angle A = m\angle C$. Using numbers, we could say that if $\angle A$ is supplementary to an angle measuring 56°, then $m\angle A = 34^\circ$. $\angle C$ is also supplementary to 56°, so it too is 34°. Therefore, $m\angle A = m\angle C$.

The proof of the Same Angles Complements Theorem is in the Review Questions. Use the proof of the Same Angles Supplements Theorem to help you.

Vertical Angles Theorem

Recall the Vertical Angles Theorem from Chapter 1. We will do a formal proof here.

Given: Lines k and m intersect.

Prove: $\angle 1 \cong \angle 3$ and $\angle 2 \cong \angle 4$

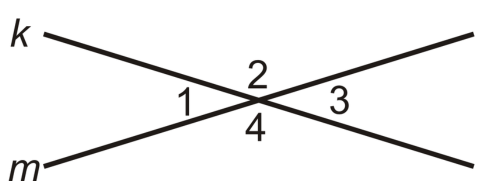


TABLE 2.22:

Statement	***Reason***
1. Lines k and m intersect	Given
2. $\angle 1$ and $\angle 2$ are a linear pair $\angle 2$ and $\angle 3$ are a linear pair $\angle 3$ and $\angle 4$ are a linear pair	Definition of a Linear Pair
3. $\angle 1$ and $\angle 2$ are supplementary $\angle 2$ and $\angle 3$ are supplementary $\angle 3$ and $\angle 4$ are supplementary	Linear Pair Postulate
4. $m\angle 1 + m\angle 2 = 180^\circ$ m $\angle 2 + m\angle 3 = 180^\circ$	

TABLE 2.22: (continued)

Statement	*Reason*
m $\angle 3 + m\angle 4 = 180^\circ$	Definition of Supplementary Angles
5. $m\angle 1 + m\angle 2 = m\angle 2 + m\angle 3$	
m $\angle 2 + m\angle 3 = m\angle 3 + m\angle 4$	Substitution PoE
6. $m\angle 1 = m\angle 3, m\angle 2 = m\angle 4$	Subtraction PoE
7. $\angle 1 \cong \angle 3, \angle 2 \cong \angle 4$	$\cong$ angles have = measures

In this proof we combined everything. You could have done two separate proofs, one for $\angle 1 \cong \angle 3$ and one for $\angle 2 \cong \angle 4$.

Example 2: In the picture $\angle 2 \cong \angle 3$ and $k \perp p$.

Each pair below is congruent. State why.

a) $\angle 1$ and $\angle 5$

b) $\angle 1$ and $\angle 4$

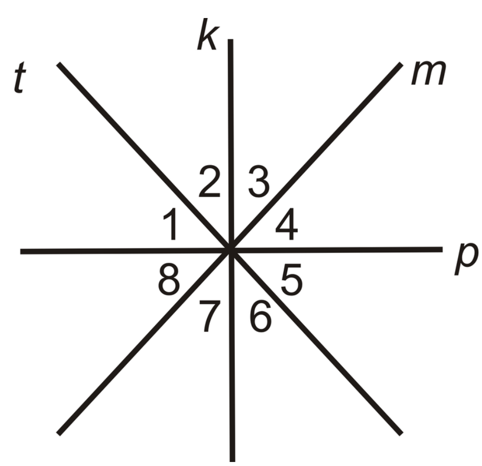
c) $\angle 2$ and $\angle 6$

d) $\angle 3$ and $\angle 7$

e) $\angle 6$ and $\angle 7$

f) $\angle 3$ and $\angle 6$

g) $\angle 4$ and $\angle 5$



Solution:

a), c) and d) Vertical Angles Theorem

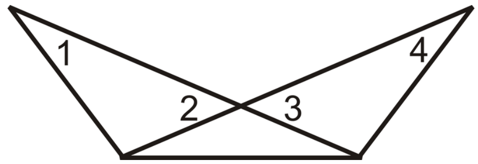
b) and g) Same Angles Complements Theorem

e) and f) Vertical Angles Theorem followed by the Transitive Property

Example 3: Write a two-column proof.

Given: $\angle 1 \cong \angle 2$ and $\angle 3 \cong \angle 4$

Prove: $\angle 1 \cong \angle 4$



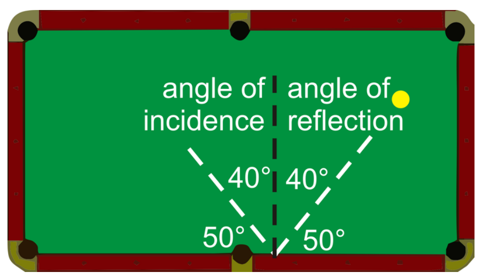
Solution:

TABLE 2.23:

Statement	*Reason*
1. $\angle 1 \cong \angle 2$ and $\angle 3 \cong \angle 4$	Given
2. $\angle 2 \cong \angle 3$	Vertical Angles Theorem
3. $\angle 1 \cong \angle 4$	Transitive PoC

Know What? Revisited If $m\angle 2 = 50^\circ$, then

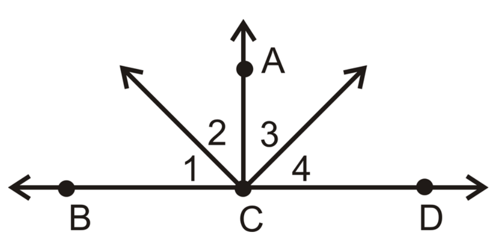
$m\angle 3 = 50^\circ$. Draw a perpendicular line at the point of reflection and the laws of reflection state that the angle of incidence is equal to the angle of reflection. So, this is a case of the Same Angles Complements Theorem. $\angle 2 \cong \angle 3$ because the angle of incidence and the angle of reflection are equal. We can also use this to find $m\angle 4$, which is 56°.



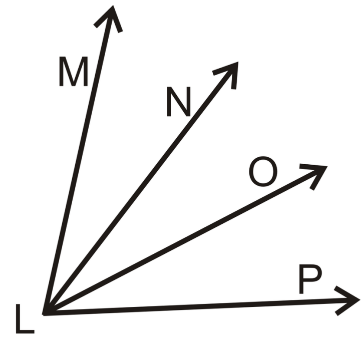
Review Questions

Write a two-column proof for questions 1-10.

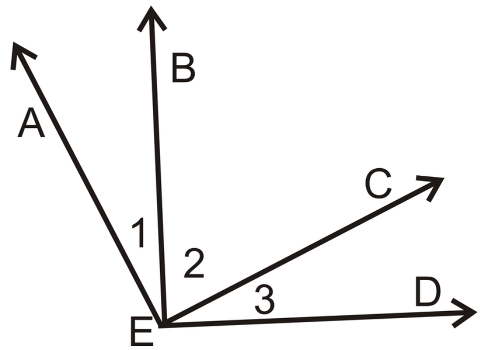
1. Given: $\overline{AC} \perp \overline{BD}$ and $\angle 1 \cong \angle 4$Prove: $\angle 2 \cong \angle 3$



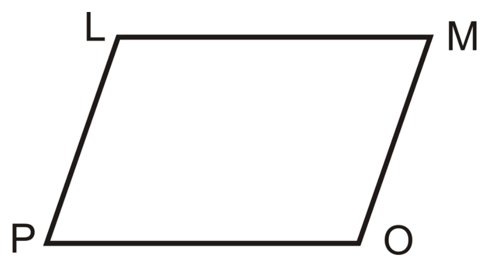
2. Given: $\angle MLN \cong \angle OLP$Prove: $\angle MLO \cong \angle NLP$



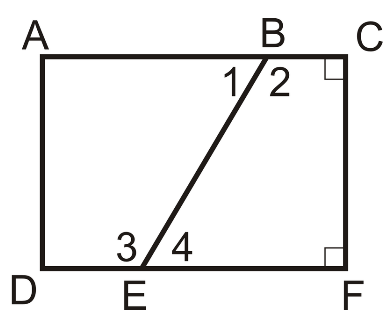
3. Given: $\overline{AE} \perp \overline{EC}$ and $\overline{BE} \perp \overline{ED}$Prove: $\angle 1 \cong \angle 3$



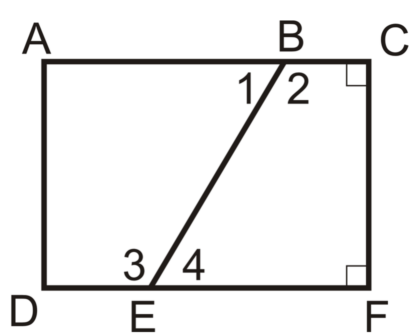
4. Given: $\angle L$ is supplementary to $\angle M \angle P$ is supplementary to $\angle O \angle L \cong \angle O$Prove: $\angle P \cong \angle M$



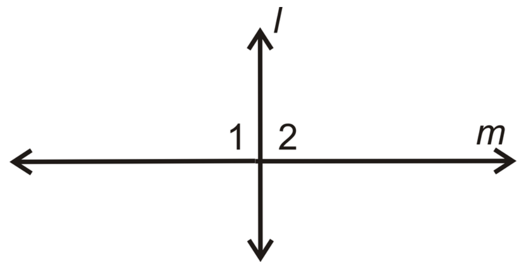
5. Given: $\angle 1 \cong \angle 4$Prove: $\angle 2 \cong \angle 3$



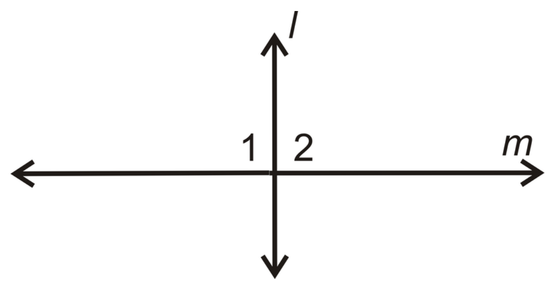
6. Given: $\angle C$ and $\angle F$ are right angles Prove: $m\angle C + m\angle F = 180^\circ$



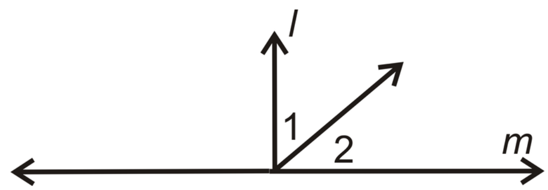
7. Given: $l \perp m$Prove: $\angle 1 \cong \angle 2$



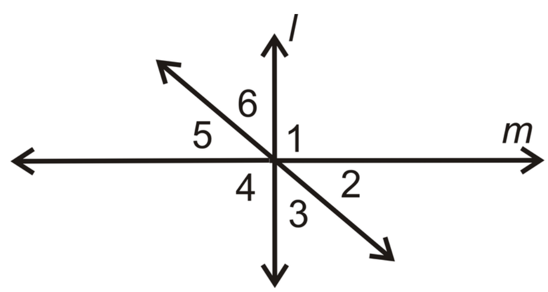
8. Given: $m\angle 1 = 90^\circ$Prove: $m\angle 2 = 90^\circ$



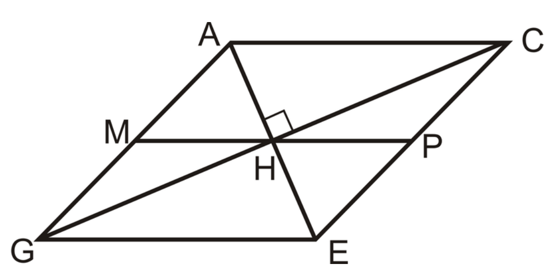
9. Given: $l \perp m$Prove: $\angle 1$ and $\angle 2$ are complements



10. Given: $l \perp m \angle 2 \cong \angle 6$Prove: $\angle 6 \cong \angle 5$



Use the picture for questions 11-20.



Given: H is the midpoint of $\overline{AE}, \overline{MP}$ and $\overline{GC}$

M is the midpoint of $\overline{GA}$

P is the midpoint of $\overline{CE}$

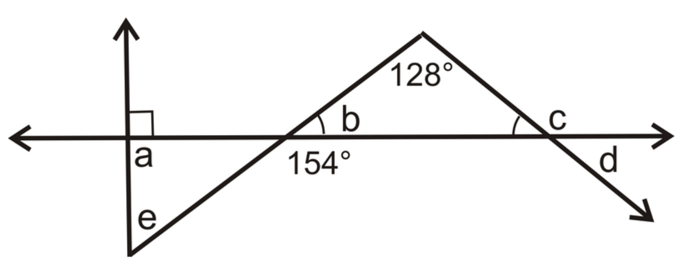
$\overline{AE} \perp \overline{GC}$

11. List two pairs of vertical angles.
12. List all the pairs of congruent segments.
13. List two linear pairs that do not have H as the vertex.
14. List a right angle.
15. List two pairs of adjacent angles that are NOT linear pairs.
16. What is the perpendicular bisector of $\overline{AE}$?
17. List two bisectors of $\overline{MP}$.
18. List a pair of complementary angles.
19. If $\overline{GC}$ is an angle bisector of $\angle AGE$, what two angles are congruent?
20. Fill in the blanks for the proof below. Given: Picture above and $\angle ACH \cong \angle ECH$ Prove: $\overline{CH}$ is the angle bisector of $\angle ACE$

TABLE 2.24:

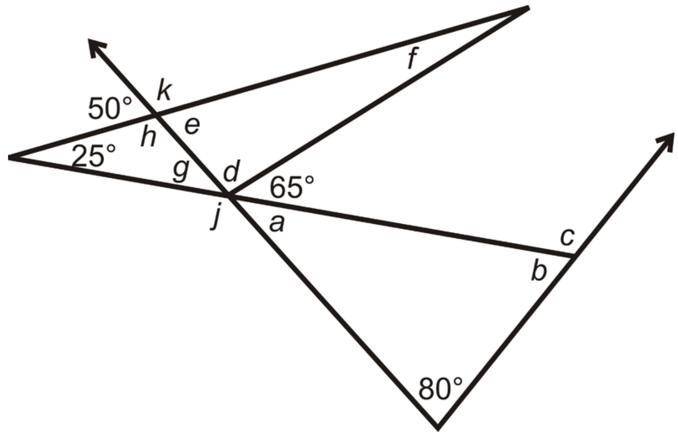
Statement	***Reason***
1. $\angle ACH \cong \angle ECH$	
$\overline{CH}$ is on the interior of $\angle ACE$	
2. $m\angle ACH = m\angle ECH$	
3.	Angle Addition Postulate
4.	Substitution
5. $m\angle ACE = 2m\angle ACH$	
6.	Division PoE
7.	

For questions 21-25, find the measure of the lettered angles in the picture below.



21. a
22. b
23. c
24. d
25. e (hint: e is complementary to b)

For questions 26-35, find the measure of the lettered angles in the picture below. *Hint: Recall the sum of the three angles in a triangle is* 180°.



26. a
27. b
28. c
29. d
30. e
31. f
32. g
33. h
34. j
35. k

Review Queue Answers

1.

TABLE 2.25:

Statement	***Reason***
1. $\overline{VX}$ is an $\angle$ bisector of $\angle WVY$ $\overline{VY}$ is an $\angle$ bisector of $\angle XVZ$	Given
2. $\angle WVX \cong \angle XVY$ $\angle XVY \cong \angle YVZ$	Definition of an angle bisector
3. $\angle WVX \cong \angle YVZ$	Transitive Property

2.6 Chapter 2 Review

Symbol Toolbox

$\rightarrow$ if-then

$\wedge$ and

$\therefore$ therefore

$\sim$ not

$\vee$ or

Keywords

Inductive Reasoning
The study of patterns and relationships is a part of mathematics. The conclusions made from looking at patterns are called **conjectures.** Looking for patterns and making conjectures is a part of **inductive reasoning,** where a rule or statement is assumed true because specific cases or examples are true.

Conjecture
The study of patterns and relationships is a part of mathematics. The conclusions made from looking at patterns are called **conjectures.**

Counterexample
We can disprove a conjecture or theory by coming up with a **counterexample.** Called proof by contradiction, only one counterexample is needed to disprove a conjecture or theory (no number of examples will prove a conjecture). The counterexample can be a drawing, statement, or number.

Conditional Statement (If-Then Statement)
Geometry uses **conditional statements** that can be symbolically written as $p \rightarrow q$ (read as "if p, then q"). "If" is the **hypothesis,** and "then" is the **conclusion.**

Hypothesis
The conditional statement is false when the hypothesis is true and the conclusion is false.

Conclusion
The second, or "then," part of a conditional statement. The conclusion is the result of a hypothesis.

Converse
A statement where the hypothesis and conclusion of a conditional statement are switched.

Inverse
A statement where the hypothesis and conclusion of a conditional statement are negated.

Contrapositive

A statement where the hypothesis and conclusion of a conditional statement are exchanged and negated.

Biconditional Statement

If $p \rightarrow q$ is true and $q \rightarrow p$ is true, it can be written as $p \rightarrow q$.
If p is not true, then we cannot conclude q is true.
If we are given q, we cannot make a conculsion. We cannot conclude p is true.

Logic

The study of reasoning.

Deductive Reasoning

Uses logic and facts to prove the relationship is always true.

Law of Detachment

The *Law of Detachment* states: If p q is true and p is true, then q is true.
If p is not true, then we cannot conclude q is true
If we are given q, we cannot make a conclusion. We cannot conclude p is true.

Law of Contrapositive

If the conditional statement is true, the converse and inverse may or may not be true. However, the contrapositive of a true statement is always true. The contrapositive is logically equivalent to the original conditional statement.

Law of Syllogism

The *Law of Syllogism* states: If $p \rightarrow q$ and $q \rightarrow r$ are true, then $p \rightarrow r$ is true.

Right Angle Theorem

If two angles are right angles, then the angles are congruent.

Same Angle Supplements Theorem

If two angles are supplementary to the same angle (or to congruent angles), then the angles are congruent.

Same Angle Complements Theorem

If two angles are complementary to the same angle (or to congruent angles), then the angles are congruent.

Reflexive Property of Equality

$a = a$.

Symmetric Property of Equality

$a = b$ and $b = a$.

Transitive Property of Equality

$a = b$ and $b = c$, then $a = c$.

Substitution Property of Equality

If $a = b$, then b can be used in place of a and vise versa.

Addition Property of Equality

If $a = b$, then $a + c = b + c$.

Subtraction Property of Equality

If $a = b$, then $a - c = b - c$.

Multiplication Property of Equality

If $a = b$, then $ac = bc$.

Division Property of Equality

If $a = b$, then $a \div c = b \div c$.

Distributive Property

$a(b + c) = ab + ac$.

Reflexive Property of Congruence

For Line Segments $\overline{AB} \cong \overline{AB}$ For Angles $\overline{AB} \cong \angle ABC \cong \angle CBA$

Symmetric Property of Congruence

For Line Segments If $\overline{AB} \cong CD$, then $\overline{CD} \cong \overline{AB}$ For Angles $\overline{CD} \cong \overline{AB}$ If $\angle ABC \cong \angle DFF \cong \angle ABC$

Transitive Property of Congruence

For Line Segments If $\overline{AB} \cong \overline{CD}$ and $\overline{CD} \cong \overline{EF}$, then $\overline{AB} \cong \overline{EF}$ For Angles If $\angle ABC \cong \angle DEF$ and $\angle DEF \cong \angle GHI$, then $\angle ABC \cong \angle GHI$

Review

Match the definition or description with the correct word.

1. $5 = x$ and $y + 4 = x$, then $5 = y + 4$ — A. Law of Contrapositive
2. An educated guess — B. Inductive Reasoning
3. $6(2a + 1) = 12a + 12$ — C. Inverse
4. 2, 4, 8, 16, 32,... — D. Transitive Property of Equality
5. $\overline{AB} \cong \overline{CD}$ and $\overline{CD} \cong \overline{AB}$ — E. Counterexample
6. $\sim p \rightarrow \sim q$ — F. Conjecture
7. Conclusions drawn from facts. — G. Deductive Reasoning
8. If I study, I will get an "A" on the test. I did not get an A. Therefore, I didn't study. — H. Distributive Property
9. $\angle A$ and $\angle B$ are right angles, therefore $\angle A \cong \angle B$. — I. Symmetric Property of Congruence
10. 2 disproves the statement: "All prime numbers are odd." — J. Right Angle Theorem — K. Definition of Right Angles

Texas Instruments Resources

In the CK-12 Texas Instruments Geometry FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9687 .